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Topological Fluid Dynamics: Theory and Applications

Recent progress in the relative equilibria of point vortices — In memoriam Hassan Aref

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Abstract

Hassan Aref, who sadly passed away in 2011, was one of the world's leading researchers in the dynamics and equilibria of point vortices. We review two problems on the subject of point vortex relative equilibria in which he was engaged at the time of his death: bilinear relative equilibria and the geometry of the three-vortex problem as it relates to equilibria. A set of point vortices is in relative equilibrium if it is at most rotating rigidly around the center of vorticity, and the configuration is bilinear if the vortices are placed on two orthogonal lines in the co-rotating frame. A very complete characterisation of the bilinear case can be obtained when one of the lines contains only two vortices. The classic three-vortex problem can be viewed anew by considering the dynamics of the circle circumscribing the vortex triangle and the interior angles of that triangle. This approach leads naturally to the observation that the equilateral triangle is the only equilibrium configuration for three point vortices, regardless of their strength values.

Selection and/or peer-review under responsibility of the Isaac Newton Institute for Mathematical Sciences, University of Cambridge

Keywords: Point vortices; relative equilibria; vortex crystals

1. Introduction

Hassan Aref passed away suddenly in his home on the 9th of September, 2011, just a few weeks before his 61st birthday. With Hassan's death the fluid dynamics community lost a great and original scientist. We have also lost a good friend, an inspiring mentor and teacher, and a prominent leader and organiser. An overview of Hassan Aref's life and work can be found in [1]. A favourite topic of Hassan was the dynamics of point vortices, and he made numerous fundamental contributions to this subject. The present paper focuses on two problems in this field that he was involved in at the time of his death: relative equilibria of point vortices arranged on perpendicular lines [2] and a geometric analysis of the three-vortex problem.

We dedicate this paper to the memory of Hassan Aref.

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2. Relative equilibria

Representing a point vortex as a complex number z, the equations of motion of N vortices moving in the velocity field they generate on each other are [3]

$$\overline{\frac{\mathrm{d}z_\alpha}{\mathrm{d}t}} = \frac{1}{2\pi\mathrm{i}} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad \alpha = 1, \ldots, N. \tag{1}$$

Here the overbar means complex conjugation and the prime on the summation sign means that $\beta = \alpha$ is excluded. The parameters Γ_α are the circulations of the vortices. It is easy to see that the complex quantity

$$Q + \mathrm{i}P = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha \tag{2}$$

is an integral of the motion. Assuming that the total circulation of the vortices is non-zero, we define the center of vorticity

$$z_{cv} = \frac{Q + \mathrm{i}P}{\sum_{\alpha=1}^{N} \Gamma_\alpha} \tag{3}$$

and choose the coordinate system such that $z_{cv} = 0$; that is, we have

$$\sum_{\alpha=1}^{N} x_\alpha = \sum_{\alpha=1}^{N} y_\alpha = 0, \tag{4}$$

where $z_\alpha = x_\alpha + \mathrm{i}y_\alpha$.

A *relative equilibrium* of the vortices is a configuration where the vortices rotate as a rigid body with constant angular velocity around the center of vorticity. In the following, we will assume that all vortices are of identical strength. Inserting $z_\alpha(t) = z_\alpha(0)\mathrm{e}^{i\Omega t}$ yields, after a suitable scaling of time, the following system of algebraic equations

$$\overline{z_\alpha} = \sum_{\beta=1}^{N}{}' \frac{1}{z_\alpha - z_\beta} \quad \alpha = 1, \ldots N. \tag{5}$$

There is a substantial body of research on the solution of these equations. A classical result by Stieltjes states that if n vortices in relative equilibrium are placed on a line, they must be located at the roots of the nth Hermite polynomial H_n. Also many solutions where the vortices are placed on concentric circles are known [4]. See [3] for a review.

While the search for analytic solutions to Eqns. (5) naturally starts with configurations with some symmetry, asymmetric configurations can be found numerically. A breakthrough was achieved by Aref & Vainchtein (1998) [5] who produced configurations with no apparent symmetry. Configurations with n vortices were found by starting with a relative equilibrium having $n - 1$ vortices of strength 1 and one vortex with very small strength ϵ at a co-rotating point, that is, at a stagnation point in the co-rotating frame. Increasing the parameter ϵ by a small amount, a new adjacent relative equilibrium is sought. If this procedure succeeds, increasing ϵ all the way to 1, a relative equilibrium with identical vortices results. Both symmetric and asymmetric configurations were found by this method.

More recently Aref & Dirksen (2011) [6] numerically found relative equilibria that are very close to being symmetric. For two examples, see Fig. 1. Numerical computations are performed with 300 digits to ensure that the asymmetric solutions are not spurious.

3. Bilinear relative equilibria

While the general problem of solving Eqns. (5) is surprisingly difficult, some progress was recently made on *bilinear equilibria*, that is, configurations where the vortices are placed on two orthogonal lines. This was the topic of

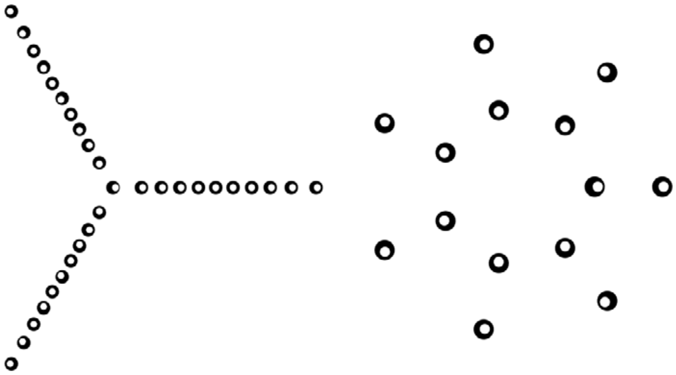

Fig. 1. Black dots show two analytical symmetric relative equilibria, with 31 and 14 vortices respectively, on regular polygons. The superimposed smaller white dots show numerically found asymmetric solutions very close to the symmetric ones. Reproduced from [6] by permission.

a paper that was submitted for publication less than two weeks before Hassan Aref's death [2]. Here we review the main results of that work.

Choosing the two lines as the real and imaginary axes of the complex plane, we consider a configuration of n vortices on the real axis at the points x_α and m vortices on the imaginary axis at the points $\mathrm{i}y_\beta$. In the following, a key role will be played by the 'generating polynomials' p and q defined by

$$p(z) = \prod_{\alpha=1}^{n}(z - x_\alpha), \quad q(z) = \prod_{\gamma=1}^{m}(z - \mathrm{i}y_\gamma). \tag{6}$$

Using Eq. (5), it can be shown that p and q fulfil the bilinear differential equation

$$pq'' + 2p'q' + p''q + 2z(pq' - p'q) + 2(n-m)pq = 0. \tag{7}$$

Conversely, if p is a polynomal of degree n with n distinct real roots and q is a polynomial of degree m with m distinct imaginary roots which fulfil Eq. (7) the roots of p and q represent a vortex system in relative equilibrium.

An example is illuminating. Let us consider a configuration with $n = 4$ vortices on the real axis and $m = 2$ vortices on the imaginary axis. From Eq. (4) it follows that the generating polynomials have the form

$$p(z) = z^4 + a_2 z^2 + a_1 z + a_0, \quad q(z) = z^2 + \eta^2, \tag{8}$$

with $\eta > 0$. Inserting these expressions in Eq. (7) and collecting terms of the same order in z yields

$$\begin{aligned}
z^4 &: \ 15 - 2\eta^2 + 2a_2 = 0,\\
z^3 &: \ a_1 = 0,\\
z^2 &: \ 3\eta^2 + 3a_2 + 2a_0 = 0,\\
z \ &: \ (\eta^2 + 3)a_1 = 0,\\
z^0 &: \ (2\eta^2 + 1)a_0 + \eta^2 a_2 = 0.
\end{aligned}$$

From the first four equations we get

$$a_1 = 0, \quad a_2 = \eta^2 - 15/2, \quad a_0 = -3\eta^2 + 45/4, \tag{9}$$

and, after some simplifications, from the last equation,

$$20\eta^4 - 48\eta^2 - 45 = 0. \tag{10}$$

This equation has one positive solution η^2, and we find

$$\eta^2 = \frac{3}{10}(4 + \sqrt{41}), \quad a_0 = \frac{9}{20}(17 - 2\sqrt{41}), \quad a_2 = \frac{1}{10}(-63 + 3\sqrt{41}). \tag{11}$$

With this, we find the following roots of p and q, corresponding to vortex positions in a relative equilibrium,

$$\pm\tfrac{1}{2}\sqrt{\tfrac{3}{5}\left[21-\sqrt{41}-\sqrt{2\left(71-\sqrt{41}\right)}\right]}\approx\pm 0.6961177525,$$
$$\pm\tfrac{1}{2}\sqrt{\tfrac{3}{5}\left[21-\sqrt{41}+\sqrt{2\left(71-\sqrt{41}\right)}\right]}\approx\pm 1.9734444009,$$
$$\pm\sqrt{\tfrac{3}{10}(4+\sqrt{41})}\,\mathrm{i}\approx\pm 1.7666174660\,\mathrm{i}.$$

The example above is a special case of a relative equilibrium where the vortices on the imaginary axis are symmetrically placed relative to the real axis. For the general case of even m we have a number of basic properties – proofs are given in [2].

Theorem 1 *Let $q(z)$ be a polynomial of even degree, m, of the form*

$$q(z)=(z^2+\eta_1^2)\ldots(z^2+\eta_{m/2}^2), \tag{12}$$

where $0<\eta_1<\ldots<\eta_{m/2}$ are given. Assume there exists a polynomial solution, $p(z)$, of the ODE

$$pq''+2p'q'+p''q+2z(pq'-p'q)+2(n-m)pq=0, \tag{13}$$

where n is a positive integer. Then

1. *$p(z)$ is of degree n*
2. *All zeros of $p(z)$ are simple, and $p(z)$ and $q(z)$ have no common zeros*
3. *$p(z)$ is an even function of z for n even, an odd function for n odd*
4. *All zeros of $p(z)$ are either real or part of a complex conjugate pair*
5. *Any other polynomial solution to Eq.* (13) *is proportional to $p(z)$*
6. *$p(0)\neq 0$ for even n*

If the generating polynomial $P(z)=p(z)q(z)$ for the total vortex system is introduced, the differential equation Eq. (7) can be rewritten in the form

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{-z^2}\frac{\mathrm{d}P}{\mathrm{d}z}\right]+r(z)P=0 \text{ where } r(z)=\mathrm{e}^{-z^2}\left[-2(n+m)+8\sum_{j=1}^{m/2}\frac{\eta_j^2}{z^2+\eta_j^2}\right]. \tag{14}$$

This has a form which allows the use of the Sturm comparison theorem, which we need in the following version:

Theorem 2 (Sturm comparison theorem) *Let L_1 and L_2 be two differential operators defined on $\mathbb{R}$ by*

$$L_1(u)=-\frac{\mathrm{d}}{\mathrm{d}z}\left[k(z)\frac{\mathrm{d}u}{\mathrm{d}z}\right]+r_1(z)u(z), \tag{15a}$$

and

$$L_2(v)=-\frac{\mathrm{d}}{\mathrm{d}z}\left[k(z)\frac{\mathrm{d}v}{\mathrm{d}z}\right]+r_2(z)v(z), \tag{15b}$$

where $k(z), r_1(z)$ and $r_2(z)$ are real-valued functions on $\mathbb{R}$, $k(z)\geq 0$, $r_1(z)$ and $r_2(z)$ are continuous, and $k(z)$ is continuously differentiable. Let x_1 and x_2 be two consecutive zeroes of a nontrivial solution, $u(z)$, of $L_1(u)=0$. If on the open interval $x_1<z<x_2$ we have $r_1(z)>r_2(z)$, then every solution $v(z)$ of $L_2(v)=0$ has a zero in this interval.

Several applications of this theorem will appear in the following. The first one is

Theorem 3 *Let $P(z)$ be a polynomial solution to Eq.* (14). *Then $P(z)$ has at least $n-m+2$ mutually distinct real zeros. In particular, for $m=2$, $P(z)$ has exactly n mutually distinct real zeros.*

Outline of proof Comparing Eqn. (14) with

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{-z^2}\frac{\mathrm{d}P_1}{\mathrm{d}z}\right]+r_1(z)P_1=0 \text{ where } r_1(z)=-2\mathrm{e}^{-z^2}(n-m) \tag{16}$$

which is the Hermite equation of order $n-m$, we have $r_1(z)>r(z)$. Letting $P_1=H_{n-m}$ be the $(n-m)$th Hermite polynomial , which is a solution to Eqn. (16) it follows from Theorem 2 that a polynomal solution P to Eqn. (14) has $n-m-1$ roots between the $n-m$ roots of H_{n-m}. A closer examination of the proof of the Sturm comparison theorem yields another two real roots of P, one above and one below the interval of roots of H_{n-m}. Finally, a parity argument using the facts that complex roots of p come in conjugate pairs according to Theorem 1, a final real root of P is found.

□

3.1. Two vortices on the imaginary axis

For $m=2$ the theorem states that the polynomial p has exactly the n distinct roots needed to ensure they represent a vortex configuration in relative equilibrium. The result is the best possible. In the next section we will discuss an example with $m=4$ and $n=5$ which has only three real zeros, and hence is a solution to Eqn. (7) but does not correspond to a vortex configuration.

We now restrict to the case $m=2$ and n arbitrary. Thus, we have

$$p(z)=a_nz^n+\cdots+a_kz^k+\cdots. \tag{17}$$

With $A=\eta^2>0$ we obtain a linear recursion relation for the a_k from the generalised Hermite equation (7),

$$A(k+2)(k+1)a_{k+2}+[(k+2)(k+1)+2A(n-k-2)]a_k+2(n-k+2)a_{k-2}=0. \tag{18}$$

Collecting the a_k in a vector $\mathbf{a}$, this can be rewritten as a matrix equation

$$\mathbf{M}_{A,n}\mathbf{a}=\mathbf{0} \tag{19}$$

where $\mathbf{M}_{A,n}$ is an $\left(\left[\frac{n}{2}\right]+1\right)\times\left(\left[\frac{n}{2}\right]+1\right)$ matrix, depending on A and n. One can show that, for fixed n,

$$|\mathbf{M}_{0,n}|=(n+2)!>0 \text{ and } |\mathbf{M}_{A,n}|\to-\infty \text{ for } A\to\infty. \tag{20}$$

By continuity it follows there exists an A such that $|\mathbf{M}_{A,n}|=0$ and hence a non-trivial coefficient vector $\mathbf{a}$ solving Eq. (19). The value of A is actually unique, which can be shown by a simple application of the Sturm comparison theorem. The polynomial $P(z)=(z^2+A)p(z)$ solves the differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(e^{-z^2}\frac{\mathrm{d}P}{\mathrm{d}z}\right)+e^{-z^2}\left[-2(n+2)+8\frac{A}{z^2+A}\right]P=0. \tag{21}$$

Assume there are solutions $P_1(z)$ for $A=A_1$ and $P_2(z)$ for $A=A_2$, $A_1>A_2>0$. Since $\dfrac{A_1}{z^2+A_1}>\dfrac{A_2}{z^2+A_2}$ for $z\neq0$ the Sturm comparison theorem applies. The polynomial P_1 has n real roots, and P_2 must then have $n-1$ roots between them. Again, an examination of the proof of the comparison theorem yields that there are further two roots of P_2 outside the interval of roots of P_1, giving in total $n+1$ real roots of P_2. But this is in contradiction with P_2 being of degree n. Thus we have

Theorem 4 *For $n=1,2,\ldots$ there is exactly one value of $\eta^2=A_n>0$ such that the differential equation*

$$(z^2+\eta^2)p''-2z(z^2+\eta^2-2)p'+2[nz^2+(n-2)\eta^2+1]p=0,$$

has a non-zero polynomial solution $p(z)$.

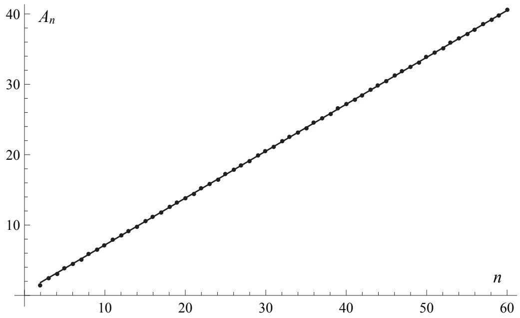

Fig. 2. Numerical determination of A determined in Theorem 4 as a function of n. The full line shows $A = \frac{2}{3}n + \frac{1}{2}$. Reproduced from [2] by permission.

Fig. 2 shows the distribution of A as a function of n from a numerical solution of $|\mathbf{M}_{A,n}| = 0$. A linear trend is very clear. Indeed, we can show

Theorem 5 *For the unique A_n determined in Theorem 4 the asymptotics is given by*

$$\alpha \equiv \lim_{n\to\infty} \frac{A_n}{n} = \frac{2}{3}. \tag{22}$$

Outline of proof Again, the Sturm comparison theorem plays a central role. The starting point is the equality

$$\frac{2}{n}\sum_{j=1}^{n}\left(2A_n/n + 4\left(x_j^{(n)}/\sqrt{2n}\right)^2\right)^{-1} + \frac{1}{2A_n} = 1 \tag{23}$$

which can be derived from Eqn. (5) without too much difficulty. Here $x_j^{(n)}$ is the jth positive root of the polynomial p. Since $A_n \to \infty$ as $n \to \infty$ it follows that

$$\lim_{n\to\infty}\frac{2}{n}\sum_{j=1}^{n}\left(2\alpha + 4\left(x_j^{(n)}/\sqrt{2n}\right)^2\right)^{-1} = 1. \tag{24}$$

It is well-known that the roots $\xi_j^{(n)}$ of the Hermite polynomial H_n are bounded by $\sqrt{2n}$ such that the normalized roots $\frac{\xi_j^{(n)}}{\sqrt{2n}}$ lie in the interval $[-1, 1]$. The basic idea is now to show that these approximate the normalized roots of p which occur in Eqn. (24) so well that we can replace them here, that is,

$$\lim_{n\to\infty}\frac{2}{n}\sum_{j=1}^{n}\left(2\alpha + 4\left(\xi_j^{(n)}/\sqrt{2n}\right)^2\right)^{-1} = 1. \tag{25}$$

This is indeed true; From the Sturm comparison theorem it is easy to see that the roots of p and H_{n+2} are interlaced. This is not quite what is needed, but with a few extra arguments the results follows.

Replacing the roots of p with roots of H_n is useful because the asymptotic density of $\xi^{(n)}/\sqrt{2n}$ is known. Calogero and Perelomov [7] have shown that it is given by

$$\rho(\xi) = \frac{4}{\pi}\sqrt{1-\xi^2} \tag{26}$$

in the sense that

$$\lim_{n\to\infty}\frac{2}{n}\sum_{j=1}^{n} f\left(\xi_j^{(n)}/\sqrt{2n}\right) = \int_{-1}^{1} f(\xi)\rho(\xi)\mathrm{d}\xi \tag{27}$$

for any continuous function f. Applying this to Eqn. (25) yields

$$\frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}}{\alpha+2\xi^2}\mathrm{d}\xi = 1 \tag{28}$$

from which one finds $\alpha = 2/3$. □

3.2. More than two vortices on the imaginary axis

For $m > 2$ Theorem 3 does not guarantee the existence of physically relevant solution $P(z)$ to Eqn. (14). Also it is not clear that one can choose values η_j^2 such that Eqn. (14) has a polynomial solution. Numerical evidence seems to indicate though that at least for $m = 4$ such values can be found. For example for $m = 4$ and $n = 5$, Eqn. (14) turns out to have a polynomial solution when (numerically) $(\eta_1^2, \eta_2^2) = (8.71216620306513, 13.204163923109789)$, with solution say $P_1(z)$ but also when $(\eta_1^2, \eta_2^2) = (2.09464882278818, 6.90535117721182)$, with solution say $P_2(z)$. Let us also write equation (14) in these cases as $L_1(P_1) = 0$ and $L_2(P_2) = 0$. Both polynomials $P_1(z)$ and $P_2(z)$ have at least $5 - 4 + 2 = 3$ real roots according to Theorem 3. However, we can use Theorem 2 and compare the differential operators L_1 and L_2. Since $P_1(z)$ has at least 3 real roots, we can conclude that $P_2(z)$ has at least 5 real roots. It turns out that $P_1(z)$ has exactly three real roots, which means that $P_1(z)$ does not give rise to a bilinear relative equilibrium of nine vortices, while $P_2(z)$ does.

Let us in general assume that for $m > 2$ and m even one can find a sequence of $m/2$-tuples $(\eta_{1,k}^2, \eta_{2,k}^2, \ldots, \eta_{m/2,k}^2)$ for $k = 1, \ldots, m/2$ fulfilling $\eta_{j,k}^2 < \eta_{j,\ell}^2$ for all $j = 1, \ldots, m/2$ and $k < \ell$. Each tuple defines a differential operator L_k from Eqn. (14), and we assume that $L_k(P) = 0$ has a polynomial solution $P_k(z)$. According to Theorem 3, $P_k(z)$ has at least $n - m + 2$ roots. However, using Theorem 2 consecutively on the operators $L_1, L_2, \ldots, L_{m/2}$, one would be able to conclude that $P_k(z)$ has at least $n - m + 2k$ real roots. Therefore the polynomial $P_{m/2}(z)$ would give rise to a physically relevant solution. We do not know if such tuples exists for any $m > 2$.

4. The geometry of an equilibrium vortex triangle

If the system under consideration consists of three vortices, one can consider an alternative formulation to (1) that is based on the geometry of the triangle with vertices at the vortex locations. Here we we give an overview of work on this topic that was in progress at the time of Hassan Aref's death.

Previous geometrical solutions focused on describing the evolution of the vortex triangle in terms of the side lengths and the enclosed area [8, 9, 10, 11]. With the three lengths defined by

$$s_1^2 = |z_2 - z_3|^2, \quad s_2^2 = |z_3 - z_1|^2, \quad s_3^2 = |z_1 - z_2|^2, \tag{29}$$

the equations governing the evolution of these sides in time are [8]

$$\frac{\mathrm{d}s_1^2}{\mathrm{d}t} = \frac{2\Delta}{\pi}\Gamma_1\frac{s_3^2 - s_2^2}{s_2^2 s_3^2}, \quad \frac{\mathrm{d}s_2^2}{\mathrm{d}t} = \frac{2\Delta}{\pi}\Gamma_2\frac{s_1^2 - s_3^2}{s_3^2 s_1^2}, \quad \frac{\mathrm{d}s_3^2}{\mathrm{d}t} = \frac{2\Delta}{\pi}\Gamma_3\frac{s_2^2 - s_1^2}{s_1^2 s_2^2}, \tag{30}$$

where the triangle area Δ is given by

$$16\Delta^2 = 2s_2^2 s_3^2 + 2s_3^2 s_1^2 + 2s_1^2 s_2^2 - s_1^4 - s_2^4 - s_3^4. \tag{31}$$

In an alternative view, the geometry of the vortex triangle can be given in terms of the interior angles and the properties of the circle that circumscribes the vortex locations [12], as shown in Fig. 3. Here we consider this formulation in examining the equilibrium configurations of three vortices when the vortex locations are not collinear. The known equilibrium configurations of this type have the vortices placed at the vertices of an equilateral triangle. The geometric analysis presented here shows in a straightforward way that the equilateral triangle is the only possible (non-collinear) equilibrium configuration for three vortices with arbitrary strengths.

Let R be the radius and $Z = X + iY$ be the center of the circumcircle passing through the vortex locations, as illustrated in Fig. 3. Then the vortex locations can be written as

$$z_1 = Z + R\mathrm{e}^{\mathrm{i}\varphi_1}, \quad z_2 = Z + R\mathrm{e}^{\mathrm{i}\varphi_2}, \quad z_3 = Z + R\mathrm{e}^{\mathrm{i}\varphi_3}, \tag{32a}$$

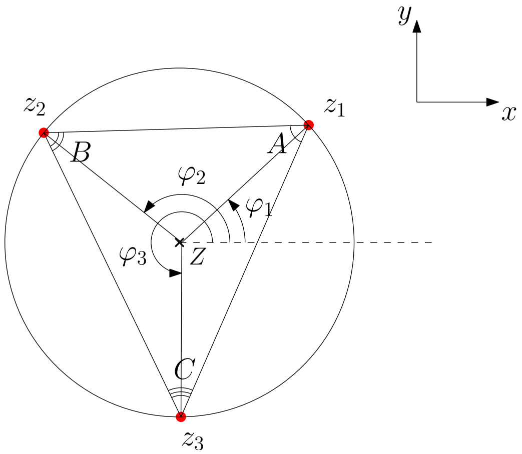

Fig. 3. Definition of the geometrical variables in the vortex triangle.

where φ_α measures the angle made by the position vector of vortex α with respect to the x (horizontal) axis. We exclude collinear configurations from this analysis, so that the vortex positions given by (32a) are well defined. We assume that the vortices are labeled anticlockwise, so that the interior angles of the triangle are given by

$$\varphi_2 - \varphi_1 = 2C, \quad \varphi_3 - \varphi_2 = 2A, \quad \varphi_1 - \varphi_3 = 2B - 2\pi. \tag{32b}$$

These interior angles are constrained by the geometry to satisfy

$$A + B + C = \pi. \tag{33}$$

These angles are related to the lengths of the triangle sides via

$$s_1 = 2R\sin A, \quad s_2 = 2R\sin B, \quad s_3 = 2R\sin C, \tag{34a}$$

and we will make use of the relation

$$R = \frac{s_1 s_2 s_3}{4|\Delta|}. \tag{34b}$$

The equation of motion for R can be obtained through manipulation of the equations of motion for the vortex positions (1). For example, substituting the notation (32) into the equation for vortex 1 gives

$$\overline{\frac{\mathrm{d}z_1}{\mathrm{d}t}} = \frac{1}{2\pi\mathrm{i}R}\left(\frac{\Gamma_2}{\mathrm{e}^{\mathrm{i}\varphi_1} - \mathrm{e}^{\mathrm{i}\varphi_2}} + \frac{\Gamma_3}{\mathrm{e}^{\mathrm{i}\varphi_1} - \mathrm{e}^{\mathrm{i}\varphi_3}}\right) = \frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi_1}}{2\pi R}\left(\frac{\Gamma_2}{1 - \mathrm{e}^{\mathrm{i}2C}} + \frac{\Gamma_3}{1 - \mathrm{e}^{-\mathrm{i}2B}}\right). \tag{35a}$$

The complex conjugate of this expression can be rewritten as

$$\dot{Z}\,\mathrm{e}^{-\mathrm{i}\varphi_1} + \dot{R} + \mathrm{i}R\dot{\varphi}_1 = \frac{1}{4\pi R}\left[\Gamma_2\cot C - \Gamma_3\cot B + \mathrm{i}(\Gamma_2 + \Gamma_3)\right], \tag{35b}$$

where the overdot denotes the time-derivative. The real component of this equation, together with the similar relations for the velocity components of vortices 2 and 3, can be written in matrix form as

$$\begin{bmatrix} \cos\varphi_1 & \sin\varphi_1 & 1 \\ \cos\varphi_2 & \sin\varphi_2 & 1 \\ \cos\varphi_3 & \sin\varphi_3 & 1 \end{bmatrix} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{R} \end{bmatrix} = \frac{1}{4\pi R}\begin{bmatrix} \Gamma_2\cot C - \Gamma_3\cot B \\ \Gamma_3\cot A - \Gamma_1\cot C \\ \Gamma_1\cot B - \Gamma_2\cot A \end{bmatrix}. \tag{35c}$$

Using the constraint (33) with (35c), one can determine the equation for $\dot{R}$ in terms of the angles A, B, C to be

$$8\pi R\frac{\mathrm{d}R}{\mathrm{d}t} = \Gamma_1\cot B\cot C(\cot B - \cot C) + \Gamma_2\cot C\cot A(\cot C - \cot A) + \Gamma_3\cot A\cot B(\cot A - \cot B). \tag{36}$$

For equilibrium configurations with finite (constant) R, the right-hand-side of (36) must be zero, giving a second constraint on the values of A, B, C independent of the value of R. Cases in which R varies with time can also be considered, and we plan to address this analysis in a subsequent publication. For equilibrium configurations with finite (constant) R, the right-hand-side of (36) must be zero, giving a second constraint on the values of A, B, C independent of the value of R.

Now consider the equations governing the time evolution of A, B, C. By differentiating (34a) we have, for example,

$$\cot A \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2R\sin A}\frac{\mathrm{d}s_1}{\mathrm{d}t} - \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{37}$$

From the equations of motion for the sides (30) we have

$$\frac{\mathrm{d}s_1}{\mathrm{d}t} = \frac{\Gamma_1\Delta}{\pi s_1}\frac{s_3^2 - s_2^2}{s_3^2 s_2^2} = \frac{\Gamma_1}{8\pi R}\frac{\cos(2B) - \cos(2C)}{\sin B \sin C} = \frac{\Gamma_1}{4\pi R}\sin A(\cot B - \cot C). \tag{38}$$

Substituting (36) and (38) into (37) gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\Gamma_1(1-\cot B\cot C)(\cot B-\cot C)-\Gamma_2\cot C\cot A(\cot C-\cot A)-\Gamma_3\cot A\cot B(\cot A-\cot B)}{8\pi R^2\cot A}, \tag{39}$$

and equivalent expressions can be obtained for the evolution of angles B and C.

For equilibrium configurations, $\dot{R} = 0$ in (37), and (39) reduces to

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\Gamma_1}{8\pi R^2}\frac{\cot B - \cot C}{\cot A}; \tag{40a}$$

the equivalent equations for $\dot{B}$ and $\dot{C}$ are

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\Gamma_2}{8\pi R^2}\frac{\cot C - \cot A}{\cot B}, \tag{40b}$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\Gamma_3}{8\pi R^2}\frac{\cot A - \cot B}{\cot C}. \tag{40c}$$

Thus, the requirement of a triangular equilibrium configuration in which $\dot{A} = \dot{B} = \dot{C} = \dot{R} = 0$, together with the constraint in (33), requires that $A = B = C = \pi/3$. This result shows that the equilateral triangle is the only equilibrium configuration of three vortices that are not collinear.

5. Conclusions

The problem of finding relative equilibrium configurations of point vortices is rich and interesting. We are far from any general theory of the structure of the solution set, even in the case of identical vortices, and there seems to be a need for new mathematical techniques. Through one of Hassan Aref's last papers [2] the Sturm comparison theorem was introduced in this topic, giving a series of rigorous results on bilinear equilibria. The hope is that this approach will yield further results. For example, we have only touched upon cases with more than $m = 2$ vortices on the imaginary axis, and perhaps something can be said about multi-linear configurations.

While the three-vortex problem is very well understood, there are still new facets worth examining. The new geometrical approach to this problem that we have discussed throws light on this classical problem from a different perspective. It allows us to obtain known equilibrium solutions in a straightforward manner, and suggests that a similar approach may be fruitful for investigating the equilibria of vortex N-gons with $N > 3$.

Indeed, point vortex dynamics in general is a wonderful place for interaction between fluid mechanics and a large and growing range of mathematical ideas. We sorely miss the insight that Hassan Aref provided in leading such investigations.

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